Growing Sheaves on fertile Categories

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Abstract

A sheaf (on a topological space) encompasses data which is attached to the open sets of some space. Well-known examples include the sheaves of continuous maps from one space to another or the structure sheaf of an affine scheme. We give a self-contained introduction to the required results of category theory before we introduce the notion of a sheaf taking values in any category. The main result states that under certain circumstances, every presheaf has a sheaf associated to it. These results are directly based on the original works of Gray (see [Gr65]).

You should note that even of this very abstract setting, there have been significant generalizations – if you are interested, [KaSch06] is a very thorough textbook. Most of the proofs of the category-theoretical results are taken from [Brc94], which is also very recommendable for further reading.

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1 Categories

Definition 1.1. A category $\mathcal{C}$ consist of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$, the **objects of** $\mathcal{C}$.
- A set $\mathcal{C}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the $\mathcal{C}$**-morphisms from** X **to** Y.
- For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a map

  $$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z) \quad , \quad (f,g) \longmapsto g \circ f$$

  which we call the **composition in** $\mathcal{C}$.

such that the following holds:

- The composition in $\mathcal{C}$ is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
- For all $X \in \mathrm{Ob}(\mathcal{C})$ there exists $\mathrm{id}_X \in \mathcal{C}(X,X)$ with the property that for all $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Z,X)$: $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

A category is **small** when $\mathrm{Ob}(\mathcal{C})$ is a set. For our purposes, a category is called **concrete** if its objects are sets, morphisms are maps between them and the composition of morphisms is precisely the composition of these maps.

Notation. We require some common terminology for dealing with categories:

- We reserve the right to write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$.
- The notation $f : X \to Y$ means $f \in \mathcal{C}(X,Y)$.

Definition 1.2. Let $\mathcal{C}$ be a category and $f : X \to Y$. Then we call f

1. an **epimorphism** if for all $h_1, h_2 : Y \to Z$, the equality $h_1 \circ f = h_2 \circ f$ already implies $h_1 = h_2$.
2. a **monomorphism** if for all $g_1, g_2 : Z \to X$, the equality $f \circ g_1 = f \circ g_2$ already implies $g_1 = g_2$.
3. an **isomorphism** if there exists a morphism $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$. We call f^{-1} the **inverse of** f.

Remark. It is easy to see that if an inverse exists, it is unique: Assuming that g and h are both inverse to f yields $h = h \circ f \circ g = g$.

Definition 1.3. We say that two objects $X, Y \in \mathcal{C}$ are **isomorphic** if there exists an isomorphism $X \to Y$. We denote this by $X \cong Y$.

Notation. We write

1. $f : X \hookrightarrow Y$ if f is a monomorphism
2. $f : X \twoheadrightarrow Y$ if f is an epimorphism
3. $f : X \xrightarrow{\sim} Y$ if f is an isomorphism.

We also say that a morphism is **epic** or **monic** when we want to say that it is an epimorphism or monomorphism, respectively. We sometiems say that epimorphisms (resp. monomorphisms) can be **right-cancelled** (resp. **left-cancelled**), which is just another way to describe their defining properties.

Definition 1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** F **from** $\mathcal{C}$ **to** $\mathcal{D}$ consists of the following data:

- For every $X \in \mathrm{Ob}(\mathcal{C})$, an object $F(X) \in \mathrm{Ob}(\mathcal{D})$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a map of sets

$$F = F_{X,Y} : \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(F(X), F(Y)).$$

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. The latter (compatibility with composition) is also being referred to as **functoriality**. We write functors as $F : \mathcal{C} \to \mathcal{D}$.

The **composition** of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is defined to be the functor $G \circ F$ which is defined by $(G \circ F)(X) = G(F(X))$ on objects and the composition $G \circ F$ (as maps of sets) on morphisms. It is obvious that this defines a functor $\mathcal{C} \to \mathcal{E}$.

Remark. Beware: Categories and functors together do not form a category, as hinted at before. However, one can consider the category **Cat** of small categories with funtors as morphisms.

Definition 1.5. Let F and G be two functors $\mathcal{C} \to \mathcal{D}$. A **natural transformation** $\varphi : F \to G$ is a class $\{ \varphi_X : F(X) \to G(X) \mid X \in \mathrm{Ob}(\mathcal{C}) \}$ of morphisms such that

$$\begin{tikzcd} F(X) \arrow[r, "\varphi_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\varphi_Y"] & G(Y) \end{tikzcd}$$

we have $G(f) \circ \varphi_X = \varphi_Y \circ F(f)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$. This property is also referred to as **naturality**.

If the φ_X are $\mathcal{D}$–isomorphisms for all $X \in \mathrm{Ob}(\mathcal{C})$, we say that φ is a **natural isomorphism** (or **natural equivalence**). We then sometimes write $F \cong_\varphi G$. We define $\mathrm{Nat}(F, G)$ to be the class of all natural transformations $F \to G$.

Definition 1.6. Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category. We denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the **functor category** where

- Objects are functors $\mathcal{C} \to \mathcal{D}$.
- Morphisms from F to G are the natural transformations $\mathrm{Nat}(F, G)$.
- Composition of natural transformations $\varphi : F \to G$ and $\psi : G \to H$ is defined by $(\psi \circ \varphi)_X := \psi_X \circ \varphi_X$ for every $X \in \mathrm{Ob}(\mathcal{C})$.

Remark. Note that $\mathcal{C}$ has to be small for this definition to make sense; otherwise, the class of natural transformations from one functor to another is not necessarily a set.

Fact 1.7. *The natural isomorphisms are the isomorphisms in* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proof. Simply note that $\varphi : F \to G$ is a natural transformation with inverse ψ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ if and only if $\psi_X \circ \varphi_X = \mathrm{id}_{F(X)}$ for all X. $\square$

2 Limits

Definition 2.1. Let $\mathcal{I}$ be a small category and $F : \mathcal{I} \to \mathcal{C}$ a functor. We say that (X, φ) is a **cone over** F when $\{ \varphi_I : X \to F(I) \mid I \in \mathcal{I} \}$ is a family of morphisms such that for all $\mathcal{I}$–morphisms $\iota : I \to J$, the equality $F(\iota) \circ \varphi_I = \varphi_J$ holds. We say that (L, λ) is a **limit of** F if any other cone (X, φ) factors uniquely through L in the sense that

$$\begin{tikzcd}
& X \arrow[d, dotted, "\exists! \mu"] \arrow[ddl, bend right, "\varphi_I"'] \arrow[ddr, bend left, "\varphi_J"] & \\
& L \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] & \\
F(I) \arrow[rr, "F(\iota)"'] & & F(J)
\end{tikzcd}$$

We often write $\lim(F) := L$ and λ^F instead of λ to denote the dependence on F. The dual notion is that of a cocone and a colimit:

$$\begin{tikzcd}
& Y & \\
& C \arrow[u, dotted, "\exists! \nu"] & \\
F(I) \arrow[ur, "\gamma_I"] \arrow[uur, bend left, "\psi_I"] \arrow[rr, "F(\iota)"'] & & F(J) \arrow[ul, "\gamma_J"'] \arrow[uul, bend right, "\psi_J"']
\end{tikzcd}$$

Again, we write $\mathrm{colim}(F) := C$ and γ^F instead of γ sometimes.

Example 2.2. Pullbacks are limits of functors from the path category of

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

whereas equalizers are limits of functors from the path category of

$$\bullet \rightrightarrows \bullet$$

Also, products indexed by some set I are limits of functors from the discrete category over I.

Lemma 2.3. *Let $F : \mathcal{I} \to \mathcal{C}$ be a functor.*

- *If (L, λ) is the limit of F, then any two morphisms $f, g : X \to L$ are equal $\Leftrightarrow$ $\lambda_I \circ f = \lambda_I \circ g$ for all $I \in \mathcal{I}$.*
- *If (C, γ) is the colimit of F, then any two morphisms $f, g : C \to Y$ are equal $\Leftrightarrow$ $f \circ \gamma_I = g \circ \gamma_I$ for all $I \in \mathcal{I}$.*

Proof. We prove only the first statement since the second follows dually (and equally trivial): Both f and g are factorizations of the cone $(X, (\lambda_I \circ f))$. $\square$

Definition 2.4. We say that a category $\mathcal{C}$ **has (co)limits of type** $\mathcal{I}$ if every functor $F : \mathcal{I} \to \mathcal{C}$ has a (co)limit. A category is said to be **(co)complete** if it has limits of type $\mathcal{I}$ for any small category $\mathcal{I}$.

Proposition 2.5. *Let $\mathcal{I}$ and $\mathcal{C}$ be small categories and $\mathcal{D}$ any category. Let $F : \mathcal{I} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We denote by $F(-)(X) : \mathcal{I} \to \mathcal{D}$ the functor defined by*

$$\begin{tikzcd} I \arrow[r, mapsto] \arrow[d, dash] & F(I)(X) \arrow[d, dash] \\ f \arrow[r, mapsto] \arrow[d] & F(f)_X \arrow[d] \\ J \arrow[r, mapsto] & F(J)(X) \end{tikzcd}$$

If $F(-)(X)$ has a limit for all $X \in \mathrm{Ob}(\mathcal{C})$, then F has a limit $(L, (\lambda_I))$ such that $(L(X), (\lambda_I^X))$ is a limit of $F(-)(X)$.

Proof. In the following, let always $f : X \to Y$ be a $\mathcal{C}$–morphism and $\iota : I \to J$ a $\mathcal{I}$–morphism. Let $(L(X), \lambda^X)$ denote the limit of $F(-)(X)$ where

$$\lambda_I^X : L(X) \to F(I)(X) \quad \text{such that} \quad \lambda_J^X = F(\iota)_X \circ \lambda_I^X. \tag{1}$$

The naturality of $F(\iota)$

$$\begin{tikzcd}
F(I)(X) \arrow[r, "F(\iota)_X"] \arrow[d, "F(I)(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(J)(X) \arrow[d, "F(J)(f)"] \\
F(I)(Y) \arrow[r, "F(\iota)_Y"'] & F(J)(Y)
\end{tikzcd}$$

implies that the maps $F(I)(f) \circ \lambda_I^X$ satisfy

$$F(\iota)_Y \circ F(I)(f) \circ \lambda_I^X = F(J)(f) \circ F(\iota)_X \circ \lambda_I^X = F(J)(f) \circ \lambda_J^X$$

and therefore constitute a cone over $F(-)(Y)$. Consequently,

$$\begin{tikzcd}
& L(X) \arrow[dl, "\lambda_I^X"'] \arrow[dr, "\lambda_J^X"] \arrow[d, dotted, "\exists! L(f)"] & \\
F(I)(X) \arrow[d, "F(I)(f)"'] & L(Y) \arrow[dl, "\lambda_I^Y"] \arrow[dr, "\lambda_J^Y"'] & F(J)(X) \arrow[d, "F(J)(f)"] \\
F(I)(Y) \arrow[rr, "F(\iota)_Y"'] & & F(J)(Y)
\end{tikzcd}$$

there exists a unique morphism $L(f) : L(X) \to L(Y)$ with the property that

$$\lambda_I^Y \circ L(f) = F(I)(f) \circ \lambda_I^X. \tag{2}$$

We now claim that L is functorial. Indeed, assume that $g : Y \to Z$ is another $\mathcal{C}$–morphism, then we have

$$\begin{aligned}
\lambda_I^Z \circ L(g \circ f) &= F(I)(g \circ f) \circ \lambda_I^X \\
&= F(I)(g) \circ F(I)(f) \circ \lambda_I^X \\
&= F(I)(g) \circ \lambda_I^Y \circ L(f) \\
&= \lambda_I^Z \circ L(g) \circ L(f)
\end{aligned}$$

and by the uniqueness of (2), this asserts functoriality. We also observe that the equality (2) means that $\lambda_I : L \to F(I)$ is a natural transformation for every $I \in \mathrm{Ob}(\mathcal{I})$ (or, look at the parallelograms in the diagram). By (1), the pair $(L, (\lambda_I))$ constitutes a cone over F.

To verify that it is a limit of F, assume that $(K, (\kappa_I))$ is another cone over F. Since the κ_I also satisfiy

$$\forall X \in \mathrm{Ob}(\mathcal{C}) : \kappa_J^X = F(\iota)_X \circ \kappa_I^X, \tag{3}$$

we get a unique morphism $\mu^X : K(X) \to L(X)$

$$\begin{tikzcd}
 & K(X) \arrow[dl, bend right, "\kappa_I^X"'] \arrow[dr, bend left, "\kappa_J^X"] \arrow[d, dotted, "\exists!\mu^X"] & \\
 & L(X) \arrow[dl, "\lambda_I^X"'] \arrow[dr, "\lambda_J^X"] & \\
F(I)(X) \arrow[rr, "F(\iota)_X"'] & & F(J)(X)
\end{tikzcd}$$

with the property that $\kappa_I^X = \lambda_I^X \circ \mu^X$ for all $I \in \mathrm{Ob}(\mathcal{I})$. Also,

$$\begin{aligned}
\lambda_I^Y \circ L(f) \circ \mu^X &\overset{(2)}{=} F(I)(f) \circ \lambda_I^X \circ \mu^X = F(I)(f) \circ \kappa_X \\
&\overset{(3)}{=} \kappa_I^Y \circ K(f) = \lambda_I^Y \circ \mu^Y \circ K(f)
\end{aligned}$$

so $L(f) \circ \mu^X = \mu^Y \circ K(f)$ by 2.3. This means that $\mu : K \to L$ is the unique natural transformation with $\kappa_I = \lambda_I \circ \mu$ – and hence, we are done. □

Corollary 2.6. *If $\mathcal{C}$ is (co-)complete, then so is* $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$. □

Definition 2.7. A **directed set** $(I, \leq)$ is a set I with a reflexive, transitive relation "$\leq$" with the additional property that every pair of elements $i, j \in I$ has an upper bound, i.e. an element $k \in I$ such that $i \leq k$ and $j \leq k$. A small category $\mathcal{I}$ with $\mathrm{Ob}(\mathcal{I}) = I$ and

$$\mathcal{I}(i,j) = \begin{cases} \{\, i \to j \,\} & ; \quad i \leq j \\ \emptyset & ; \quad \text{otherwise} \end{cases}$$

for some directed set $(I, \leq)$ is then called a **directed category**.

Let $\mathcal{I}$ be a directed category. A functor $F : \mathcal{I} \to \mathcal{C}$ is called a **directed functor**. It can be represented as a family of objects $X_i = F(i)$ and morphisms $f_{ij} = F(i \to j)$. In this case, we say that (X_i, f_{ij}) is a **directed diagram in $\mathcal{C}$**. A colimit of type $\mathcal{I}$ is called a **direct limit**. We then write

$$\varinjlim X_i := \mathrm{colim}(F)$$

Dually, a functor $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ is given by a family of objects $Y_i = G(i)$ and morphisms $g_{ij} = F(i \leftarrow j)$. We say that (Y_i, g_{ij}) is an **inverse diagram in $\mathcal{C}$**. A limit of type $\mathcal{I}^{\mathrm{op}}$ is called an **inverse limit** and we write

$$\varprojlim Y_i := \lim(F)$$

Fact 2.8. *Inside a category $\mathcal{C}$, assume that*

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\times"] & X \arrow[d, "g"] \\ Y \arrow[r, "f"'] & S \end{tikzcd}$$

is a pullback diagram. If g is a monomorphism (resp. isomorphism), then so is g'. In other words, any pullback of a monomorphism (resp. isomoprhims) is monic (resp. an isomorphism).

Proof. Assume that g is monic and let $u, v : Q \to P$ be two morphisms such that $g' \circ u = g' \circ v$. We define $g'' := g' \circ u$ and $f'' := f' \circ u$. Thus,

$$f \circ g'' = f \circ g' \circ u = g \circ f' \circ u = g \circ f''$$

means that (Q, f'', g'') is a cone. Therefore, u is the unique morphism that makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[dr, "u"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ & Y \arrow[r, "f"'] & S \end{tikzcd}$$

On the other hand, we also have $g' \circ v = g' \circ u = g''$. From

$$g \circ f' \circ v = f \circ g' \circ v = f \circ g' \circ u = f \circ g'' = g \circ f''$$

we can cancel g (it is monic) to obtain $f' \circ v = f'' = f' \circ u$ – and by uniqueness of u, this implies $u = v$.

Let us assume now that g is an isomorphism. It is then easy to see that $P := Y$, $f' := g^{-1} \circ f$ and $g' := \mathrm{id}_Y$ is a pullback, and any two pullbacks differ by unique isomorphism (which will then be g'). $\square$

Fact/Definition 2.9. The **kernel pair** of a $\mathcal{C}$–morphism $f : X \to Y$ is a triple (K, α, β) with

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\times"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

if the above pullback diagram exists.

Fact 2.10. *For $f : X \to Y$, the following conditions are equivalent:*

1. *f is a monomorphism.*
2. *The kernel pair of f exists and is given by $(X, \mathrm{id}_X, \mathrm{id}_X)$.*
3. *The kernel pair (K, α, β) of f exists and $\alpha = \beta$.*

Proof. For the implication (1) $\Rightarrow$ (2), it suffices to verify the universal property of the cone $(X, \mathrm{id}_X, \mathrm{id}_X)$: Given two morphisms $u, v : Z \to X$ with $f \circ u = f \circ v$, since f is monic, the unique factorization is given by $u = v$. The implication (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), let $u, v : Z \to X$ be such that $f \circ u = f \circ v$. Then, there exists a unique $h : Z \to K$ such that $u = \alpha \circ h = \beta \circ h = v$. □

3 Intersections and Generators

Definition 3.1. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$ an object. On the (possibly proper) class $\{\, (U, u) \mid u : U \hookrightarrow X \,\}$, we define a binary relation by

$$(U, u) \leq (V, v) :\Leftrightarrow u \text{ factors through } v \Leftrightarrow \exists \nu : U \to V : u = v \circ \nu$$

It is easy to see that this is a partial order. We continue to define an equivalence relation

$$(U, u) \sim (V, v) :\Leftrightarrow u \leq v \text{ and } v \leq u$$

The equivalence classes with respect to $\sim$ are called the **subobjects of** X, denoted by $\mathrm{Sub}(X)$.

Fact 3.2. *Let X be an object and $(U, u) \sim (V, v)$ two representatives of the same subobject of X. Then, U and V are isomorphic via the factorization of u through v (and v through u).*

Proof. By definition, v factors through u via some $\nu : V \hookrightarrow U$ which has to be monic since both u and v are monic. Equivalently, u factors through v via some $\mu : U \hookrightarrow V$, so

$$\begin{tikzcd}
U \arrow[rr, hook, "u"] \arrow[dr, hook, shift left, "\mu"] & & X \\
& V \arrow[ul, hook, shift left, "\nu"] \arrow[ur, hook', "v"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

Now $u \circ \nu \circ \mu = v \circ \mu = u = u \circ \mathrm{id}_U$ and u can be left-cancelled, so we have $\nu \circ \mu = \mathrm{id}_U$ and symmetrically, we get $\mu \circ \nu = \mathrm{id}_V$. □

Notation 3.3. For ease of notation, we mean by $U \in \mathrm{Sub}(X)$ the equivalence class of a pair denoted by (U, ι_U). This notation is slightly abusive because there might be more than one monomorphism $U \hookrightarrow X$. However, we shall never abuse it to imply the contrary.

Definition 3.4. A category $\mathcal{C}$ is called **well-powered** when $\mathrm{Sub}(X)$ is a set for any object $X \in \mathrm{Ob}(\mathcal{C})$.

Example 3.5. Clearly, the category **Sets** is well-powered: The subobjects of any set are in bijection with its subsets. Similarly, the categories **Grp** and **Ab** are well-powered.

Definition 3.6. Let X be an object of some category $\mathcal{C}$. The **intersection** of any family of subobjects of $\mathfrak{U} \subseteq \mathrm{Sub}(X)$ is defined to be its infimum in $\mathrm{Sub}(X)$, if it exists. We denote this by

$$\bigcap \mathfrak{U} := \inf_{\mathrm{Sub}(X)} (\mathfrak{U}).$$

Furthermore, the **union** of $\mathfrak{U}$ is defined to be its supremum

$$\bigcup \mathfrak{U} := \sup_{\mathrm{Sub}(X)} (\mathfrak{U}).$$

In the case $\mathfrak{U} = \{ U, V \}$, we write $U \cap V := \bigcap \mathfrak{U}$ as well as $U \cup V := \bigcup \mathfrak{U}$ and similarly for all finite sets $\mathfrak{U}$.

Fact 3.7. *Let X be an object of a category $\mathcal{C}$ such that $\mathrm{Sub}(X)$ is a set. Then, the intersection of any family of subobjects of X exists if and only if the union of any family of subobjects of X exists.*

Proof. This follows because

$$\begin{aligned} \inf(\mathfrak{U}) &= \sup \{ V \mid \forall U \in \mathfrak{U} : V \leq U \} \\ \sup(\mathfrak{U}) &= \inf \{ V \mid \forall U \in \mathfrak{U} : V \geq U \} \end{aligned}$$

holds in any partially ordered set. □

Proposition 3.8. *Let U and V be two subobjects of X. If the pullback of the diagram $U \xhookrightarrow{\iota_U} X \xhookleftarrow{\iota_V} V$ exists, it is the intersection $U \cap V$.*

Proof. We consider the pullback diagram

$$\begin{tikzcd} W \arrow[r, hook, "\lambda_U"] \arrow[d, hook, "\lambda_V"'] \arrow[dr, phantom, "\times"] & U \arrow[d, hook, "\iota_U"] \\ V \arrow[r, hook, "\iota_V"'] & X \end{tikzcd}$$

where all morphisms are monic by 2.8. Now this means that

$$\iota_U \circ \lambda_U = \iota_V \circ \lambda_V =: \iota_W$$

define the same monomorphism and W is a subobject of X which is smaller than both U and V. On the other hand, given any subobject W' which is smaller than U and V, we obtain a cone that factors through W – so indeed, $W = U \cap V$. □

Proposition 3.9. *In a complete and well-powered category $\mathcal{C}$, the intersection of any family of subobjects of some object $X \in \mathrm{Ob}(\mathcal{C})$ always exists.*

Proof. Let S be a complete system of representatives for a subclass of $\mathrm{Sub}(X)$. Let $\mathcal{I}$ be the subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{I}) = \{ U \mid (U, \iota_U) \in S \}$ and

$$\mathcal{I}(U,V) := \begin{cases} \{ \iota_U \} & ; \quad V = X \\ \{ \mathrm{id}_U \} & ; \quad V = U \\ \emptyset & ; \quad \text{otherwise} \end{cases}$$

Then, we can consider the limit (L, γ) of the inclusion functor $\mathcal{I} \to \mathcal{C}$. We claim that $\gamma_X : L \to X$ is a monomorphism. Indeed, if $u, v : T \to L$ are morphisms with $\gamma_X \circ u = \gamma_X \circ v$, then we have

$$\iota_U \circ \gamma_U \circ u = \gamma_X \circ u = \gamma_X \circ v = \iota_U \circ \gamma_U \circ v$$

for all $U \in S$. Now since ι_U is monic for all such U, we can conclude that $\gamma_U \circ u = \gamma_U \circ v$ holds for all $U \in S$. But then, by 2.3, we know that $u = v$. Therefore, (L, γ_X) defines a subobject of X. By construction of hte limit, γ_X factors through any subobject given by $U \in S$. □

Notation 3.10. The following notational concept is slightly abusive and will only be used in this subsection. Let $\mathcal{C}$ be a category. If $\mathfrak{G} \subseteq \mathrm{Ob}(\mathcal{C})$ is a subclasses of objects, we define

$$\mathcal{C}(\mathfrak{G}, X) := \bigcup_{G \in \mathfrak{G}} \mathcal{C}(G, X)$$

We then write $f : \mathfrak{G} \to X$ to mean $f \in \mathcal{C}(\mathfrak{G}, X)$.

Definition 3.11. Let $\mathcal{C}$ be a category. A subclass $\mathfrak{G} \subseteq \mathrm{Ob}(\mathcal{C})$ of objects is called a **family of generators of** $\mathcal{C}$ if $\mathfrak{G}$ is a set and the following condition holds for any two $\mathcal{C}$–morphisms $u, v : X \to Y$:

$$(u \neq v) \Rightarrow (\exists g : \mathfrak{G} \to X : u \circ g \neq v \circ g).$$

If $\mathfrak{G} = \{G\}$ consists of a single element, we say that G is a **generator of** $\mathcal{C}$.

Fact 3.12. *If $\mathfrak{G}$ is a generating family for $\mathcal{C}$, then a $\mathcal{C}$–morphism $f : X \to Y$ is monic if and only if the following cndition holds: For all $u, v : \mathfrak{G} \to X$, the equality $f \circ u = f \circ v$ implies $u = v$.*

Proof. Let $x, y : T \to X$ be two arbitrary morphisms with $f \circ x = f \circ y$. Now if x were different from y, then there would be a $g : \mathfrak{G} \to T$ with the property that $x \circ g \neq y \circ g$. But on the other hand, $f \circ x \circ g = f \circ y \circ g$ means $x \circ g = y \circ g$ by assumption, which is absurd. Hence, $x = y$. □

Example 3.13. Let us give some examples of generators.

- In **Sets**, a generator is given by the one-point-set $\{*\}$.
- In the category $\mathbf{Alg}_R$ of R–algebras, $R[X]$ is a generator.
- In the category $\mathbf{Mod}_R$ of R–modules, R itself is a generator.
- In the category **Grp**, the group $(\mathbb{Z}, +)$ is a generator.

The above list of examples motivates the following definition:

Fact/Definition 3.14. Let $\mathcal{C}$ be a concrete category. A **free generator in one variable** is an object $G \in \mathrm{Ob}(\mathcal{C})$ with an element $g \in G$ and the property that

$$\forall T \in \mathrm{Ob}(\mathcal{C}) : \forall t \in T : \exists!\, g_t : G \to T \text{ such that } g_t(g) = t.$$

In other words, there is a unique morphism $G \to T$ for every object T which is defined by the image of g. We sometimes write $\langle g \rangle$ for the object G to denote this. We claim that a free generator is a generator in the sense of 3.11.

Proof. Let $u, v : X \to Y$ be two different morphisms. Then, there exists some $x \in X$ such that $v(x) \neq u(x)$. Let $g : G \to X$ be such that $g(x_i) = x$ for all i. Then, $v(g(x_1)) \neq u(g(x_1))$ and therefore $v \circ g \neq u \circ g$. □

Definition 3.15. A family $\mathfrak{G}$ of objects of $\mathcal{C}$ is said to be a **family of strong generators** when the following condition holds: If $\iota : U \hookrightarrow X$ is a monomorphism which is not an isomorphism, there exists a $g : \mathfrak{G} \to X$ which does not factor through ι (there exists no $\bar{g}$ with $g = \iota \circ \bar{g}$). In the case $\mathfrak{G} = \{G\}$, we say that G is a **strong generator**.

We note that, in general, the two notions of generator and strong generator do not coincide. However, the following observations show that under relatively humble conditions, they do.

Fact 3.16. *If $\mathcal{C}$ has equalizers, then any family of strong generators is a family of generators.*

Proof. Let $\mathfrak{G}$ be a family of strong generators. Assume that we have two morphisms $u, v : X \to Y$ with $u \neq v$. Then there exists some $g : \mathfrak{G} \to X$ which does not factor through the equalizer $\mathrm{eq}(u, v) \hookrightarrow X$ of u and v. Consequently, $u \circ g \neq v \circ g$. □

Fact 3.17. *If every $\mathcal{C}$–monomorphism is the equalizer of a pair of morphisms, then every family of generators is a family of strong generators.*

Proof. Let $\mathfrak{G}$ be a family of generators. Let $\iota : U \hookrightarrow X$ be a proper monomorphism. By assumption, there exist $\alpha, \beta : X \to Y$ such that $U = \mathrm{eq}(\alpha, \beta)$. We know that $\alpha \neq \beta$ because otherwise, ι would be an isomorphism. Hence, there has to be a $G \in \mathfrak{G}$ and $g : G \to X$ with $g \circ \alpha \neq g \circ \beta$. This means that g does not factor through $U = \mathrm{eq}(\alpha, \beta)$. □

Remark 3.18. Note that this particularly implies that all the generators in 3.13 are strong.

Proposition 3.19. *If $\mathcal{C}$ is a category with finite limits that has a strong family $\mathfrak{G}$ of generators, then $\mathcal{C}$ is well-powered.*

Proof. Let $X \in \mathcal{C}$ be an object. Consider the set

$$\mathfrak{S} := \coprod_{G \in \mathfrak{G}} \mathcal{C}(G, C) = \{ (G, g) \mid G \in \mathfrak{G},\, g : G \to C \}$$

Now, we associate to each $U \in \mathrm{Sub}(X)$ the subset

$$\sigma_U = \{ (G, g) \mid \exists h : G \to U : \iota_U \circ h = g \} .$$

Assume now that $V, U \in \mathrm{Sub}(X)$ such that $\sigma_U = \sigma_V$. We consider the following pullback diagram (see also 3.8):

$$\begin{tikzcd} U \cap V \arrow[r, hook, "\lambda_U"] \arrow[d, hook, "\lambda_V"'] \arrow[dr, phantom, "\times"] & U \arrow[d, hook, "\iota_U"] \\ V \arrow[r, hook, "\iota_V"'] & X \end{tikzcd}$$

It is our goal to show that λ_U is an isomorphism - by symmetry, we can then conclude that λ_V is also an isomorphism and therefore, $U = V$ (as subobjects of X). Consequentially, we can identify $\mathrm{Sub}(X)$ with a subset of the powerset of $\mathfrak{S}$, which is a set.

If λ_U was no isomorphism, there would exist some $x : G \to U$ which does not factor through $U \cap V$ by assumption on $\mathfrak{G}$. Since $g := \iota_U \circ x \in \sigma_U = \sigma_V$, there would be a $y : G \to V$ such that $\iota_V \circ y = g$ and hence, the universal property of the pullback

$$\begin{tikzcd}
G \arrow[drr, bend left, "x"] \arrow[dr, dotted, "\exists! z"] \arrow[ddr, bend right, "y"'] & & \\
& U \cap V \arrow[r, hook, "\lambda_U"] \arrow[d, hook, "\lambda_V"] \arrow[dr, phantom, "\times"] & U \arrow[d, hook, "\iota_U"] \\
& V \arrow[r, hook, "\iota_V"] & X
\end{tikzcd}$$

contradicts the assumption. Thus, λ_U is an isomorphism. □

Lemma 3.20. *Let $\mathcal{C}$ be a category with pullbacks and $\mathcal{I}$ a small one. Let $\varphi : F \to G$ be a natural transformation of functors $F, G \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. Then, φ is monic if and only if φ_I is monic for all $I \in \mathrm{Ob}(\mathcal{I})$.*

Proof. If all φ_I are monic, then φ is monic because natural transformations are composed component-wise. For the converse, assume that φ is monic. Then, we know that the difference kernel of φ is given by $(F, \mathrm{id}_F, \mathrm{id}_F)$ – recall 2.10. By 2.5, this means that $(F(I), \mathrm{id}_{F(I)}, \mathrm{id}_{F(I)})$ is difference kernel of φ_I and again by 2.10, φ_I is monic. □

Corollary 3.21. *If a category $\mathcal{C}$ has finite limits and a family of strong generators, then $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is well-powered for any small category $\mathcal{I}$.*

Proof. By 3.19, $\mathcal{C}$ is well-powered. If $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$, the assignment

$$\begin{array}{ccc}
\mathrm{Sub}(F) & \longrightarrow & \prod_{I \in \mathrm{Ob}(\mathcal{I})} \mathrm{Sub}(F(I)) \\
(G, \varphi) & \longmapsto & (G(I), \varphi_I)_I
\end{array}$$

is well-defined by 3.20. To see that it defines an injective map of sets (and therefore proving the statement), let us assume that $(G(I), \varphi_I) \sim (H(I), \psi_I)$. Then, we have isomorhisms $\mu_I : G(I) \to H(I)$ with

$$\begin{tikzcd}
G(I) \arrow[rr, hook, "\varphi_I"] \arrow[dr, "\mu_I"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & F(I) \\
& H(I) \arrow[ur, hook, "\psi_I"'] &
\end{tikzcd}$$

For any $\mathcal{I}$–morphism $\iota : I \to J$, we can compute

$$\psi_J \circ \mu_J \circ G(\iota) = \varphi_J \circ G(\iota) = F(\iota) \circ \varphi_I = F(\iota) \circ \psi_I \circ \mu_I = \psi_J \circ H(\iota) \circ \mu_I$$

by applying (in this order) the property of μ, naturality of φ, property of μ and naturality of ψ. Since ψ_J is monic, we can cancel it to verify that $\mu := (\mu_I)$ is a natural transformation with $\varphi = \psi \circ \mu$. Since μ is a natural isomorphism, $(G, \varphi) \sim (H, \psi)$ and we are done. $\square$

4 Sheaves

Definition 4.1. Let X be any topological space. We can then consider the directed category $\mathcal{T}_X$ where

- objects are the open subsets $U \mathring{\subseteq} X$.
- relations are given by $U \leq V :\Leftrightarrow U \supseteq V$.

For every subset $M \subseteq X$, we denote by $\mathfrak{U}_X(M)$ the full subcategory of $\mathcal{T}_X$ induced by all open neighbourhoods of M (e.g. all open sets U with $U \supseteq M$). We often write $\mathfrak{U}(M)$ instead of $\mathfrak{U}_X(M)$ if there is no ambiguity concerning X. We also set $\mathfrak{U}_X(p) := \mathfrak{U}_X(\{p\})$.

An **open cover** $\mathcal{U}$ of X is a set of open subsets of X such that $\bigcup \mathcal{U} = X$. An open cover is called **strong** if it is closed under finite intersections – i.e. $U, V \in \mathcal{U}$ implies that $U \cap V \in \mathcal{U}$.

Definition 4.2. Let $\mathcal{C}$ be a category and X a topological space.

$$\mathbf{PreSh}(X, \mathcal{C}) := \mathrm{Fun}(\mathcal{T}_X, \mathcal{C})$$

The **category of $\mathcal{C}$–presheaves on** X. A presheaf $\mathscr{F}$ is called a **sheaf** if for every strong covering $\mathcal{V}$ of any open subset $U \mathring{\subseteq} X$,

$$\mathscr{F}(U) = \varprojlim_{V \in \mathcal{V}} \mathscr{F}(V).$$

This defines the full subcategory $\mathbf{Sh}(X, \mathcal{C})$ of $\mathbf{PreSh}(X, \mathcal{C})$ which we call the **category of $\mathcal{C}$–sheaves on** X.

Proposition 4.3. *The inclusion functor $\mathbf{Sh}(X, \mathcal{C}) \hookrightarrow \mathbf{PreSh}(X, \mathcal{C})$ reflects all limits. In other words, whenever a diagram of sheaves has a limit, then this limit is a sheaf.*

Proof. Let $F : \mathcal{I} \to \mathbf{PreSh}(X, \mathcal{C}) = \mathrm{Fun}(\mathcal{T}_X, \mathcal{C})$ be a functor which maps every object $I \in \mathrm{Ob}(\mathcal{I})$ to a sheaf $\mathscr{F}_I := F(I)$. Assume that F has a limit $(\mathscr{L}, \lambda)$. Let $\mathcal{V}$ be a strong covering of some $U \mathring{\subseteq} X$. The above then means that

$$\lambda^I : \mathscr{L} \to \mathscr{F}_I \text{ with } F(\iota) \circ \lambda^I = \lambda^J \text{ for all } \iota \in \mathcal{I}(I, J). \tag{4}$$

We now need to show that $(\mathscr{L}(U), \rho^{\mathscr{L}}_{U/-})$ is a limit for $\mathscr{L}|_{\mathcal{V}}$. Assume that (T, τ) is a cone over $\mathscr{L}|_{\mathcal{V}}$, i.e.

$$\tau_W = \rho^{\mathscr{L}}_{V/W} \circ \tau_V \text{ for all } V, W \in \mathcal{V} \text{ such that } W \subseteq V.$$

Since λ^I is a morphism of presheaves and $\mathscr{F}_I(U) = \varprojlim_{V \in \mathcal{V}} \mathscr{F}(V)$ for every $I \in \mathrm{Ob}(\mathcal{C})$, we obtain

$$\begin{tikzcd}
 & T \arrow[dl, "\tau_V"'] \arrow[dr, "\tau_W"] \arrow[dd, dotted, "\exists!\delta^I"] & \\
\mathscr{L}(V) \arrow[rr, "\rho^{\mathscr{L}}_{V/W}"] \arrow[dd, "\lambda^I_V"'] & & \mathscr{L}(W) \arrow[dd, "\lambda^I_W"] \\
 & \mathscr{F}_I(U) \arrow[dl, "\rho^I_{U/V}"'] \arrow[dr, "\rho^I_{U/W}"] & \\
\mathscr{F}_I(V) \arrow[rr, "\rho^I_{V/W}"'] & & \mathscr{F}_I(W)
\end{tikzcd} \tag{5}$$

where we write $\rho^I_{V/W} := \rho^{\mathscr{F}_I}_{V/W}$. If $\iota : I \to J$ is a $\mathcal{I}$–morphism, we notice that for all $V \in \mathcal{V}$,

$$\begin{aligned}
\rho^J_{U/V} \circ F(\iota)_U \circ \delta^I &= F(\iota)_V \circ \rho^I_{U/V} \circ \delta^I \\
&= F(\iota)_V \circ \lambda^I_V \circ \tau_V \\
&= \lambda^J_V \circ \tau_V \\
&= \rho^J_{U/V} \circ \delta^J
\end{aligned}$$

implying $F(\iota)_U \circ \delta^I = \delta^J$ by 2.3. Because $(\mathscr{L}(U), \lambda^-_U)$ is a limit for $F(-)(U)$ by 2.5,

$$\begin{tikzcd}
 & T \arrow[d, dotted, "\exists!\tau_U"] \arrow[ddl, bend right, "\delta^I"'] \arrow[ddr, bend left, "\delta^J"] & \\
 & \mathscr{L}(U) \arrow[dl, "\lambda^I_U"'] \arrow[dr, "\lambda^J_U"] & \\
\mathscr{F}_I(U) \arrow[rr, "F(\iota)_U"'] & & \mathscr{F}_J(U)
\end{tikzcd}$$

In particular, this implies that

$$\lambda^I_V \circ \rho^{\mathscr{L}}_{U/V} \circ \tau_U = \rho^I_{U/V} \circ \delta^I = \lambda^I_V \circ \tau_V$$

so again by 2.3, we know that $\rho^{\mathscr{L}}_{U/V} \circ \tau_U = \tau_V$ for all $V \in \mathcal{V}$. In diagram form, this means

$$\begin{tikzcd}
 & T \arrow[d, "\tau_U"] \arrow[ddl, bend right, "\tau_V"'] \arrow[ddr, bend left, "\tau_W"] & \\
 & \mathscr{L}(U) \arrow[dl, "\rho^{\mathscr{L}}_{U/V}"'] \arrow[dr, "\rho^{\mathscr{L}}_{U/W}"] & \\
\mathscr{L}(V) \arrow[rr, "\rho^{\mathscr{L}}_{V/W}"'] & & \mathscr{L}(W)
\end{tikzcd}$$

for $V \subseteq W$ and $V, W \in \mathcal{V}$. We are left to verify that τ_U is unique with this property.

Now, if $\theta : T \to \mathscr{L}(U)$ is another morphism such that $\rho^{\mathscr{L}}_{U/V} \circ \theta = \tau_V$ for all $V \in \mathcal{V}$, then we fix some $I \in \mathrm{Ob}(\mathcal{I})$ and use the equalities from (5) to calculate that

$$\rho^I_{U/V} \circ \delta^I = \lambda^I_V \circ \tau_V = \lambda^I_V \circ \rho^{\mathscr{L}}_{U/V} \circ \theta = \rho^I_{U/V} \circ \lambda^I_U \circ \theta.$$

for all $V \in \mathcal{V}$. By 2.3, this means $\lambda^I_U \circ \theta = \delta^I$ and therefore, $\theta = \tau_U$ since τ_U was unique with precisely this property. $\square$

Proposition/Definition 4.4. Assume that $\mathcal{C}$ has products. For an open covering $\mathcal{U} = \{ U_i \mid i \in I \}$ of any open subset $U \mathring{\subseteq} X$, consider the diagram

$$\mathscr{F}(U) \xrightarrow{\alpha^{\mathscr{F}}_{\mathcal{U}}} \prod_{i \in I} \mathscr{F}(U_i) \underset{\breve{\omega}^{\mathscr{F}}_{\mathcal{U}}}{\overset{\hat{\omega}^{\mathscr{F}}_{\mathcal{U}}}{\rightrightarrows}} \prod_{i,j \in I} \mathscr{F}(U_{ij}) \tag{6}$$

where the morphisms are defined as

$$\alpha^{\mathscr{F}}_{\mathcal{U}} = \prod_{i \in I} \rho^{\mathscr{F}}_{U/U_i} \qquad \hat{\omega}^{\mathscr{F}}_{\mathcal{U}} = \prod_{i,j \in I} \rho^{\mathscr{F}}_{U_i/U_{ij}} \circ \pi_i \qquad \breve{\omega}^{\mathscr{F}}_{\mathcal{U}} = \prod_{i,j \in I} \rho^{\mathscr{F}}_{U_j/U_{ij}} \circ \pi_j.$$

It is called the **sheaf sequence** of $\mathscr{F}$. We often write $\alpha_{\mathcal{U}}$, $\alpha^{\mathscr{F}}$ or simply α instead of $\alpha^{\mathscr{F}}_{\mathcal{U}}$ if the sheaf and/or the covering are clear from the context. The same holds for $\hat{\omega}$ and $\breve{\omega}$.

Now, we claim that $\mathscr{F}$ is a sheaf if and only if $(\mathscr{F}(U), \alpha^{\mathscr{F}}_{\mathcal{U}})$ is the equalizer of $\omega^{\mathscr{F}}_{\mathcal{U}}$ and $\breve{\omega}^{\mathscr{F}}_{\mathcal{U}}$ for all open coverings $\mathcal{U}$ of U.

Remark. Note that we always have

$$\begin{aligned}
\hat{\omega}^{\mathscr{F}}_{\mathcal{U}} \circ \alpha^{\mathscr{F}}_{\mathcal{U}} &= \textstyle\prod_{ij} \rho^{\mathscr{F}}_{U_i/U_{ij}} \circ \rho^{\mathscr{F}}_{U/U_i} = \prod_{ij} \rho^{\mathscr{F}}_{U/U_{ij}} \\
&= \textstyle\prod_{ij} \rho^{\mathscr{F}}_{U_j/U_{ij}} \circ \rho^{\mathscr{F}}_{U/U_j} = \breve{\omega}^{\mathscr{F}}_{\mathcal{U}} \circ \alpha^{\mathscr{F}}_{\mathcal{U}}.
\end{aligned}$$

by the presheaf property. It therefore suffices to check the universal property of the equalizer if one wants to verify for a certain presheaf $\mathscr{F}$ that it is a sheaf.

Proof. First, we note that every open cover $\mathcal{U} = \{ U_i \mid i \in I \}$ gives rise to a strong open cover $\overline{\mathcal{U}} := \bigcup_{n\in\mathbb{N}} \{ U_{\bar{\imath}} \mid \bar{\imath} \in I^n \}$ Thus, if $\mathscr{F}$ is a sheaf and such an open cover $\mathcal{U}$ is given, then

$$\mathscr{F}(U) = \varprojlim_{V\in\overline{\mathcal{U}}} \mathscr{F}(V).$$

Let $\varphi : T \to \prod_i \mathscr{F}(U_i)$ be some morphism such that $\hat{\omega} \circ \varphi = \check{\omega} \circ \varphi$. We define maps $\varphi_{\bar{\imath}} : T \to \mathscr{F}(U_{\bar{\imath}})$ for all $U_{\bar{\imath}} \in \overline{\mathcal{U}}$ as follows: For $i \in I$, we have $\varphi_i := \pi_i \circ \varphi$. Then, set $\varphi_{\bar{\imath}} := \rho_{U_{i_0}/U_{\bar{\imath}}} \circ \varphi_{i_0}$ for any multiindex $\bar{\imath} = (i_0, \dots, i_n)$. We now claim that these define a cone. Hence, assume that we have another multiindex $\bar{\jmath} = (j_0, \dots, j_m)$ such that $U_{\bar{\imath}} \leq U_{\bar{\jmath}}$, i.e.

$$U_{i_0} \cap \dots \cap U_{i_n} \subseteq U_{j_0} \cap \dots \cap U_{j_m} \subseteq U_{j_0}.$$

Our goal is to show that $\rho_{U_{\bar{\jmath}}/U_{\bar{\imath}}} \circ \varphi_{\bar{\jmath}} = \varphi_{\bar{\imath}}$. Let us first assume $\bar{\jmath} = (j)$. For $\bar{\imath} = (i)$, the statement follows from our assumption on φ since $U_i \subseteq U_j$ means $U_i = U_{ij}$. We use this as a base case for an induction to prove that

$$\begin{aligned}
\rho_{U_j/U_{\bar{\imath}}} \circ \varphi_j &= \rho_{U_j/U_{i_0\cdots i_n}} \circ \varphi_j \\
&= \rho_{U_{i_0\cdots i_{n-1}}/U_{i_0\cdots i_n}} \circ \rho_{U_j/U_{i_0\cdots i_{n-1}}} \circ \varphi_j \\
&= \rho_{U_{i_0\cdots i_{n-1}}/U_{i_0\cdots i_n}} \circ \varphi_{i_0\cdots i_{n-1}} \\
&= \rho_{U_{i_0\cdots i_{n-1}}/U_{i_0\cdots i_n}} \circ \rho_{U_{i_0}/U_{i_0\cdots i_{n-1}}} \circ \varphi_{i_0} \\
&= \rho_{U_{i_0}/U_{i_0\cdots i_n}} \circ \varphi_{i_0} \\
&= \varphi_{i_0\cdots i_n} = \varphi_{\bar{\imath}}
\end{aligned}$$

Now we let $\bar{\jmath}$ be of arbitrary length and calculate

$$\rho_{U_{\bar{\jmath}}/U_{\bar{\imath}}} \circ \varphi_{\bar{\jmath}} = \rho_{U_{\bar{\jmath}}/U_{\bar{\imath}}} \circ \rho_{U_{j_0}/U_{\bar{\jmath}}} \circ \varphi_{j_0} = \rho_{U_{j_0}/U_{\bar{\imath}}} \circ \varphi_{j_0} = \varphi_{\bar{\imath}}.$$

This now means that φ uniquely factorizes through the limit $\mathscr{F}(U)$ verifying that $\mathscr{F}(U)$ is an equalizer for $\hat{\omega}$ and $\check{\omega}$.

Conversely, let $\mathcal{U} = \{ U_i \mid i \in I \}$ be a strong covering of $U \overset{\circ}{\subseteq} X$. Whenever $i, j \in I$ are indices, let (ij) be an index such that $U_{(ij)} = U_i \cap U_j$. Now, assume that $\varphi_i : T \to \mathscr{F}(U_i)$ is a family of morphisms satisfying $\varphi_i = \rho_{U_j/U_i} \circ \varphi_j$ whenever $U_i \subseteq U_j$. We then have to show that φ factors through $\mathscr{F}(U)$. Now, $\mathcal{U}$ is a covering of U – and by assumption, $\mathscr{F}(U)$ is the equalizer of the corresponding $\hat{\omega}$ and $\check{\omega}$. Let $\varphi : T \to \prod_i \mathscr{F}(U_i)$ the morphism induced by the φ_i. Then,

$$\hat{\omega} \circ \varphi = \prod\nolimits_{i,j} \rho_{U_i/U_{ij}} \circ \varphi_i = \prod\nolimits_{i,j} \varphi_{(ij)} = \prod\nolimits_{i,j} \rho_{U_j/U_{ij}} \circ \varphi_j = \check{\omega} \circ \varphi$$

means that φ has to factor through $\mathscr{F}(U)$ as required. $\square$

Corollary/Definition 4.5. Let $\mathcal{C}$ be a preadditive category with products and $\mathscr{F}$ a $\mathcal{C}$–presheaf on X. If $\mathcal{U} = \{ U_i \mid i \in I \}$ is an open cover for $U \mathring{\subseteq} X$, the diagram

$$\mathscr{F}(U) \xrightarrow{\alpha_{\mathcal{U}}^{\mathscr{F}}} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\omega_{\mathcal{U}}^{\mathscr{F}}} \prod_{i,j \in I} \mathscr{F}(U_{ij}) \tag{7}$$

with $\omega = \hat{\omega} - \check{\omega}$ is called the **additive sheaf sequence**. By the above, $\mathscr{F}$ is a sheaf if and only if $(\mathscr{F}(U), \alpha_{\mathcal{U}}^{\mathscr{F}}) = \ker(\omega_{\mathcal{U}}^{\mathscr{F}})$. $\square$

Example 4.6. If $\mathscr{F}$ is the presheaf

$$\mathscr{F}(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous } \}$$

with restriction of maps, we check (7). Given an open cover $\mathcal{U} = \{ U_i \mid i \in I \}$ and morphisms $f_i : U_i \to \mathbb{R}$ with $\omega((f_i)_i) = 0$, this means that

$$\begin{aligned} 0 = \omega\left((f_i)_i\right) &= \left(\textstyle\prod_{ij} \rho_{U_i/U_{ij}} \circ \pi_i - \rho_{U_j/U_{ij}} \circ \pi_j\right)\left((f_i)_i\right) \\ &= \left(\rho_{U_i/U_{ij}}(f_i) - \rho_{U_j/U_{ij}}(f_j)\right) = \left(f_i|_{U_{ij}} - f_j|_{U_{ij}}\right) \end{aligned}$$

or, in other words, the maps agree on all intersections. We can therefore glue them to a continuous map $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$, so $\alpha(f) = \left(f|_{U_i}\right)$. In other words, the presheaf of continuous $\mathbb{R}$–valued functions on X is a sheaf.

Note that if we replace "continuous" by "bounded", this is no longer true – the glueing of locally bounded functions must not be bounded.

Definition 4.7. Let $\mathscr{F}$ be a $\mathcal{C}$–presheaf on X and $p \in X$. If it exists, the **stalk of** $\mathscr{F}$ **in** p is defined to be the direct limit

$$\mathscr{F}_p := \varinjlim_{U \in \mathfrak{U}(p)} \mathscr{F}(U).$$

For every $U \in \mathfrak{U}(p)$, we denote by $\sigma_{U/p}^{\mathscr{F}} : \mathscr{F}(U) \to \mathscr{F}_p$ the canonical morphisms of this limit. We sometimes write $\sigma_{U/p}$ when there is no ambiguity concerning the presheaf $\mathscr{F}$. If the stalk exists for all $p \in X$, we say that $\mathscr{F}$ **has stalks**.

Fact/Definition 4.8. For any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of $\mathcal{C}$–presheaves on X which have a stalk at $p \in X$, the morphisms $\sigma_{U/p}^{\mathscr{G}} \circ \varphi_U$ constitute a cocone. We then define $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ to be the unique morphism such that

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\sigma_{U/p}^{\mathscr{F}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{G}(U) \arrow[d, "\sigma_{U/p}^{\mathscr{G}}"] \\ \mathscr{F}_p \arrow[r, dotted, "\varphi_p"'] & \mathscr{G}_p \end{tikzcd}$$

Proof. For any $V \subseteq U$, we have

$$\sigma^{\mathscr{G}}_{U/p} \circ \varphi_U = \sigma^{\mathscr{G}}_{V/p} \circ \rho^{\mathscr{G}}_{U/V} \circ \varphi_U = \sigma^{\mathscr{G}}_{V/p} \circ \varphi_V \circ \rho^{\mathscr{F}}_{U/V}. \qquad \square$$

Exercise 4.9. If $\mathcal{C}$ has limits of type $\mathfrak{U}(p)$ for some $p \in X$, check that $(-)_p$ defines a functor $(-)_p : \mathbf{Sh}(X, \mathcal{C}) \longrightarrow \mathcal{C}$.

5 Sheafification

For the rest of this subsection, let X be a topological space and $\mathcal{C}$ a category. We will therefore write **PreSh** instead of $\mathbf{PreSh}(X, \mathcal{C})$ and **Sh** instead of $\mathbf{Sh}(X, \mathcal{C})$.

Fact/Definition 5.1. Assume that $\mathcal{C}$ has direct limits and products. There is a functor

$$(-)_+ : \mathbf{PreSh} \to \mathbf{Sh} \quad \text{with} \quad \mathscr{F}_+(U) = \prod_{p \in U} \mathscr{F}_p$$

and restriction maps are just the projections. We call it the **stalkification functor**. There are morphisms $\sigma^{\mathscr{F}} : \mathscr{F} \to \mathscr{F}_+$ which are natural in $\mathscr{F}$ and given by the formula

$$\sigma^{\mathscr{F}}_U = \prod_{p \in U} \sigma^{\mathscr{F}}_{U/p}.$$

We call $\sigma^{\mathscr{F}}$ the **stalkification morphism of** $\mathscr{F}$.

Proof. It is immediate that $\mathscr{F}_+$ is a presheaf. The fact that it is a sheaf then follows from the universal property of the product.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, the morphisms φ_p (see 4.8) give rise to a morphism $\varphi_+ : \mathscr{F}_+ \to \mathscr{G}_+$ defined by $(\varphi_+)_U := \prod_{p \in U} \varphi_p$. Compatibility with the restriction is obvious since those are just the projections, so we are left to check functoriality. Given another morphism $\psi : \mathscr{G} \to \mathscr{H}$, this follows simply because $(-)_p$ is a functor.

The fact that $\sigma^{\mathscr{F}}$ is a morphism of presheaves again just follows from the fact that the restriction morphisms of $\mathscr{F}_+$ are the canonical projections of the products, so we are left to verify that $\sigma^{\mathscr{F}}$ is natural in $\mathscr{F}$. In other words, we need to show that

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\sigma^{\mathscr{F}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{G}(U) \arrow[d, "\sigma^{\mathscr{G}}"] \\
\prod_{p \in U} \mathscr{F}_p \arrow[r, "\prod_{p \in U} \varphi_p"'] & \prod_{p \in U} \mathscr{G}_p
\end{tikzcd}$$

for all $U \mathring{\subseteq} X$. This follows directly from 4.8:

$$\sigma^{\mathscr{G}} \circ \varphi_U = \prod_{p \in U} \sigma^{\mathscr{G}}_{U/p} \circ \varphi_U = \prod_{p \in U} \varphi_p \circ \sigma^{\mathscr{F}}_{U/p} = \prod_{p \in U} \varphi_p \circ \sigma^{\mathscr{F}} \qquad \square$$

Definition 5.2. Let $\mathcal{C}$ be a category with direct limits. An object $S \in \mathrm{Ob}(\mathcal{C})$ is called **small** if the following condition holds for all directed diagrams (X_i, ξ_{ij}) in $\mathcal{C}$ with colimit (C, γ):

Whenever there are two morphisms $f_i : S \to X_i$ and $f_j : S \to X_j$ with the property that $\gamma_i \circ f_i = \gamma_j \circ f_j$, then there is a $k \geq i, j$ such that $\xi_{ik} \circ f_i = \xi_{jk} \circ f_j$.

$$\begin{tikzcd}
 & S \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\
X_i \arrow[r, "\gamma_i"] \arrow[dr, dotted, "\exists\, \xi_{ik}"'] & C & X_j \arrow[l, "\gamma_j"'] \arrow[dl, dotted, "\exists\, \xi_{jk}"] \\
 & X_k &
\end{tikzcd}$$

If $\mathcal{C}$ has a family of strong generators which consists of small objects, we say that $\mathcal{C}$ **has enough small objects**.

Definition 5.3. Let $\mathscr{F}$ be a presheaf. The **sheafification of** $\mathscr{F}$, if it exists, is the reflection $(\mathscr{F}^+, \theta^{\mathscr{F}})$ of $\mathscr{F}$ along the inclusion functor **Sh** $\hookrightarrow$ **PreSh**.

Remark. In other words, the sheafification of $\mathscr{F}$ consists of a sheaf $\mathscr{F}^+$ and a morphism $\theta^{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^+$ of presheaves such that, for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$,

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\varphi"] \arrow[dr, "\theta^{\mathscr{F}}"'] & & \mathscr{G} \\
 & \mathscr{F}^+ \arrow[ur, dotted, "\exists!\psi"'] &
\end{tikzcd}$$

there exists a unique $\psi : \mathscr{F}^+ \to \mathscr{G}$ with $\varphi = \psi \circ \theta^{\mathscr{F}} = \varphi$.

Proposition 5.4. *If $\mathcal{C}$ has products, direct limits and enough small objects, the stalkification morphism of any sheaf is monic.*

Proof. Let $\mathscr{F}$ be a sheaf and $\sigma : \mathscr{F} \to \mathscr{F}_+$ the stalkification morphism. We omit all superscripts for the sake of readability. By 3.20, we have to show that $\sigma_U : \mathscr{F}(U) \to \mathscr{F}_+(U)$ is a monomorphism for all open subsets $U \mathring{\subseteq} X$. Hence, assume that $u, v : S \to \mathscr{F}(U)$ are morphisms with $\sigma_U \circ u = \sigma_U \circ v$. By 3.12 and our assumptions on $\mathcal{C}$, we may assume S to be a small, strong generator. Let $p \in U$ be any point, then 5.1 tells us that $\sigma_{U/p} \circ u = \sigma_{U/p} \circ v$.

Since S is small, this means that there exists some $U_p \in \mathfrak{U}_U(p)$ with the property that $\rho_{U/U_p} \circ u = \rho_{U/U_p} \circ v$. Now $\mathcal{U} := \{ U_p \mid p \in U \}$ is an open cover. Recalling (6), we get that

$$\alpha_{\mathcal{U}} \circ u = \prod_{p \in U} \rho_{U/U_p} \circ u = \prod_{p \in U} \rho_{U/U_p} \circ v = \alpha_{\mathcal{U}} \circ v$$

and because $\alpha_{\mathcal{U}}$ is monic for every open cover $\mathcal{U}$, we can conclude $u = v$. ☐

Theorem 5.5 (Existence of the Sheafification)**.** *Let $\mathcal{C}$ be a complete category with direct limits and enough small objects. Then, the sheafification $\mathscr{F}^+$ of any $\mathcal{C}$–presheaf $\mathscr{F}$ on X exists and*

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\sigma^{\mathscr{F}}"] \arrow[dr, "\theta^{\mathscr{F}}"'] \arrow[rr, phantom, shift right=6, "\circlearrowleft"] & & \mathscr{F}_+ \\
& \mathscr{F}^+ \arrow[ur, "v^{\mathscr{F}}"'] &
\end{tikzcd}$$

In other words, the unique $v^{\mathscr{F}}$ making the above commutative is monic.

Proof. Let $\mathscr{F}$ be a presheaf and let $\sigma^{\mathscr{F}} : \mathscr{F} \to \mathscr{F}_+$ be its stalkification morphism. By 3.21, we know that **PreSh** is well-powered. By 2.6, it is also complete. Thus by 3.9, the intersection of any family of subobjects of $\mathscr{F}_+$ exists. Let

$$\mathfrak{U} := \left\{ (\mathscr{G}, \gamma) \in \mathrm{Sub}(\mathscr{F}_+) \;\middle|\; \mathscr{G} \in \mathbf{Sh},\ \exists\, \theta : \mathscr{F} \to \mathscr{G} : \gamma \circ \theta = \sigma^{\mathscr{F}} \right\}$$

be the family of subsheaves of $\mathscr{F}_+$ through which the stalkification factors. By 5.1, this set is never empty. We can therefore pick $(\mathscr{F}^+, v^{\mathscr{F}}) := \bigcap \mathfrak{U}$ and define $\theta^{\mathscr{F}}$ to be the morphism such that $v^{\mathscr{F}} \circ \theta^{\mathscr{F}} = \sigma^{\mathscr{F}}$ (which exists by definition of $\mathfrak{U}$). Now, let us check that this defines a sheafification of $\mathscr{F}$. Hence, assume that $\mathscr{G}$ be a sheaf and $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism of presheaves. Consider

$$\begin{tikzcd}
\mathscr{F} \arrow[dr, dotted, "\exists!\mu"] \arrow[drr, bend left, "\sigma^{\mathscr{F}}"] \arrow[ddr, bend right, "\varphi"'] & & \\
& \mathscr{P} \arrow[r, hook, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\times"] & \mathscr{F}_+ \arrow[d, "\varphi_+"] \\
& \mathscr{G} \arrow[r, hook, "\sigma^{\mathscr{G}}"'] & \mathscr{G}_+
\end{tikzcd}$$

Since pullbacks are limits, $\mathscr{P}$ is a sheaf by 4.3. Note that β is monic by 2.8 because $\sigma^{\mathscr{G}}$ is the stalkfication morphism of $\mathscr{G}$, which is monic by 5.4. In other words, $(\mathscr{P}, \beta) \in \mathrm{Sub}(\mathscr{F}_+)$. The fact that $\sigma^{\mathscr{G}} \circ \varphi = \varphi_+ \circ \sigma^{\mathscr{F}}$ is just the naturality of σ by 5.1 and the thereby induced morphism μ proves that $(\mathscr{P}, \beta) \in \mathfrak{U}$. By minimality,

$$\begin{tikzcd}
\mathscr{F}^+ \arrow[rr, hook, "\upsilon^{\mathscr{F}}"] \arrow[dr, dotted, "\exists\kappa"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathscr{F}_+ \arrow[dr, "\varphi_+"] & \\
 & \mathscr{P} \arrow[ur, hook, "\beta"'] \arrow[dr, "\alpha"] \arrow[d, phantom, "\circlearrowleft"] & \times & \mathscr{G}_+ \\
\mathscr{F} \arrow[uu, "\theta^{\mathscr{F}}"] \arrow[ur, "\mu"] \arrow[rr, "\varphi"'] & {} & \mathscr{G} \arrow[ur, "\sigma^{\mathscr{G}}"'] &
\end{tikzcd}$$

The left triangle also commutes since β is monic and therefore,

$$\beta \circ \kappa \circ \theta^{\mathscr{F}} = \upsilon^{\mathscr{F}} \circ \theta^{\mathscr{F}} = \sigma^{\mathscr{F}} = \beta \circ \mu \quad \Longrightarrow \quad \kappa \circ \theta^{\mathscr{F}} = \mu.$$

We define $\psi := \alpha \circ \kappa$ and get that

$$\psi \circ \theta^{\mathscr{F}} = \alpha \circ \kappa \circ \theta^{\mathscr{F}} = \alpha \circ \mu = \varphi.$$

We are left to show that ψ is unique with this property. If $\psi' : \mathscr{F}^+ \to \mathscr{G}$ is a morphism with $\psi' \circ \theta^{\mathscr{F}} = \varphi$, let $(\mathscr{E}, \iota)$ be the equalizer of ψ and ψ'. Again by 4.3, $\mathscr{E}$ is a sheaf and since $\psi \circ \theta^{\mathscr{F}} = \psi' \circ \theta^{\mathscr{F}}$, there is a unique morphism $\gamma : \mathscr{F} \to \mathscr{E}$ such that $\iota \circ \gamma = \theta^{\mathscr{F}}$.

$$\begin{tikzcd}
\mathscr{E} \arrow[r, hook, "\iota"] & \mathscr{F}^+ \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\psi'"'] & \mathscr{G} \\
 & \mathscr{F} \arrow[u, "\theta^{\mathscr{F}}"'] \arrow[ul, dotted, "\exists!\gamma"] &
\end{tikzcd}$$

By minimality of $\mathscr{F}^+$, we know that ι has to be an isomorphism. This means $\psi = \psi'$. $\square$

Definition 5.6. A category $\mathcal{C}$ is called **fertile** if it is complete, has direct limits and enough small objects.

Corollary 5.7. *Let $\mathcal{C}$ be a fertile category. Then there is a left adjoint*

$$(-)^+ \dashv i : \mathbf{Sh} \leftrightarrows \mathbf{PreSh}$$

In particular,

- *The categories* **PreSh** *and* **Sh** *are complete.*
- *If $\mathcal{C}$ has comlimits of type $\mathcal{I}$, then so do* **PreSh** *and* **Sh**. *The colimit of a diagram $\mathscr{D}$ in* **Sh** *is the sheafification of the colimit of $\mathscr{D}$ in* **PreSh**.

Proposition 5.8. *The categories* **Sets**, **Grp**, $\mathbf{Alg}_R$ *and* $\mathbf{Mod}_R$ *are fertile.*

Proof. Note that all these categories are complete and cocomplete, so we are left to verify that they have enough small objects. Given a directed diagram (X_i, ξ_{ij}) indexed by $(I, \leq)$, we consider the set

$$X := \{\, (x, i) \mid i \in I,\ x \in X_i \,\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(x, i) \sim (y, j) :\Leftrightarrow \exists k \geq i, j : \xi_{ik}(x) = \xi_{jk}(y). \tag{8}$$

Let us denote the equivalence class of (x, i) by $[x, i]$. We claim that X is an object of our category. For **Sets**, this is obvious. Assume that we require a group law (denoted by $+$). We define it by choosing an index $(ij) \in I$ such that $(ij) \geq i$ and $(ij) \geq j$. We then set

$$[x, i] + [y, j] := [\xi_{i,(ij)}(x) + \xi_{j,(ij)}(y), (ij)].$$

This is well-defined: If $[y, j] = [z, k]$, we pick some $m \in I$ with the property that $m \geq ((ij)(ik))$ and $\xi_{jm}(y) = \xi_{km}(z)$. Then,

$$\begin{aligned}
[x, i] + [y, j] &= [\xi_{i,(ij)}(x) + \xi_{j,(ij)}(y), (ij)] \\
&= [\xi_{(ij),m}(\xi_{i,(ij)}(x) + \xi_{j,(ij)}(y)), m] \\
&= [\xi_{im}(x) + \xi_{jm}(y), m] \\
&= [\xi_{im}(x) + \xi_{km}(z), m] \\
&= [\xi_{i,(ik)}(x) + \xi_{j,(ik)}(z), (ik)] \\
&= [x, i] + [z, k]
\end{aligned}$$

It inherits all properties from the group laws on the X_i by definition. In $\mathbf{Alg}_R$, we also define the multiplicative group law in the above way and distributivity follows for the same reason. In $\mathbf{Alg}_R$ and $\mathbf{Mod}_R$, we define a scalar multiplication by

$$\alpha \cdot [x, i] := [\alpha \cdot x, i],$$

which is well-defined for all the same reasons as before. By definition of the algebraic structure, the maps

$$\begin{aligned}
\gamma_i : X_i &\longrightarrow X \\
x &\longmapsto [x, i]
\end{aligned}$$

are homomorphisms. Since $\gamma_j(\xi_{ij}(x)) = \gamma_i(x)$ is equivalent to (8), we can conclude that (X, γ) is a direct limit for the diagram (X_i, ξ_{ij}).

Recall from 3.13 that each of the above categories has free generator $\langle g \rangle$ in one variable. Given any two morphisms $f_i : \langle g \rangle \to X_i$ and $f_j : \langle g \rangle \to X_j$ with the property that $\gamma_i \circ f_i = \gamma_j \circ f_j$, let $x := f_i(g)$ and $y := f_j(g)$. Then, $[x, i] = [y, j]$ means that there exists some k with $\xi_{ik}(x) = \xi_{jk}(y)$ and thus, $\xi_{ik} \circ f_i = \xi_{jk} \circ f_j$. $\square$

Corollary 5.9. $\mathbf{Rings} = \mathbf{Alg}_{\mathbb{Z}}$ *and* $\mathbf{Ab} = \mathbf{Mod}_{\mathbb{Z}}$ *are fertile.* $\square$

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